International Journal of Theoretical Physics, Vol. 46, No. 5, May 2007 (© 2006)
DOI: 10.1007/s10773-006-9282-1

Hydrodynamics in 5-Dimensional Cosmological Special Relativity

Gianluca Gemelli[1]

Received August 9, 2006; accepted August 31, 2006
Published Online: December 2 2006

The dynamics of a perfect fluid is studied in 5-dimensional special relativity, a framework which can be considered the 5-d generalization of cosmological special relativity as well as the flat specialization of 5-d brane world theory. This picture, as described in an earlier paper, directly includes a particle production mechanism. Here it is showed that the source of particle production vanishes if the fluid is isentropic. Moreover it is showed that the hydrodynamical equations can be interpreted in terms of a scale factor, giving rise to a set of equations which simulate in a sense Friedmann cosmology.

KEY WORDS: 5 dymensions; relativistic hydrodynamics; cosmological relativity; friedmann equations.

1. INTRODUCTION

In a recent paper (Gemelli, 2006) a 5-dimensional generalization of the equations of special relativistic hydrodynamics was introduced. It was showed that a particle production phenomenon arises in a natural way in such framework. The 5-d cosmological special-relativity or brane world theory interpretation, in conformity with Carmeli's theories (Carmeli, 1995, 1996, 2002), with the receding velocity of galaxies playing the role of fifth dimension, is rather natural. In cosmological relativity, however, the fifth dimension is timelike, while as for the particle production mechanism described in Gemelli (2006), also a spacelike fifth dimension can be considered as well.

Particle production is a feature of recent relativistic inflationary cosmology (see e.g. Cissoko, 1998). Cosmological particle production is thought to account for negative pressure, which arises in the modelling of the accelerating universe (de Campos, 2002). The cosmological scenario with particle production is usually called open system cosmology (Prigogine *et al.*, 1989).

Then there is a possible link between cosmological relativity (or rather, the 5-d version of it) and open system cosmology, a link that in principle

[1] L. S. B. Pascal, V. P. Nenni 48, Pomezia (Roma), Italy; e-mail: gianluca.gemelli@poste.it.

permits to interpret particle production phenomena as cosmological effects. This is interesting, since in cosmology particle production is usually introduced by hand; the 5-d hydrodynamics approach instead has it as a natural consequence of 5-d cosmology, and has it in the limit of negligible gravitation.

A concrete calculation of the source of particle production was not carried out in Gemelli (2006), but, as a possible valid strategy, it was proposed to proceed to some kind of generalization of the classic thermodynamical principle:

$$T dS = d\mathcal{E} - \frac{p}{r^2} dr \tag{1}$$

where T denotes temperature, S entropy, p pressure and r matter density (baryon number). The problem was however left open in Gemelli (2006).

Here we instead see that there is no need to introduce new thermodynamical principles: it is possible in fact to obtain the expression of the source of particle production in ordinary thermodynamical terms. Thus one finds out that such source vanishes in the particular case the fluid is isentropic (Section 3).

The aim of this paper is also to see to what extent the dynamics of the mass-energy content of the Universe can be described in special relativistic terms: i.e. by considering "test" mass-energy distributions in a flat 5-d spacetime. In fact we are going to see that the 5-d cosmological special relativistic hydrodynamical equations lead to dynamical equations in terms of a scale factor, equations which resemble or are analogues in a sense to those of Friedmann's (Section 4).

Friedmann equations are (Carmeli, 2002, p. 168–169 and Wesson, 1999, p. 15):

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{1}{3}(\chi\rho_F + \Lambda) - \frac{k}{R^2} \tag{2}$$

$$\frac{\ddot{R}}{R} = -\frac{\chi}{6}(\rho_F + 3p_F) + \frac{\Lambda}{3} \tag{3}$$

where R is the scale factor, χ is the gravitational constant, Λ is the cosmological constant, $k = +1, 0, -1$ and ρ_F and p_F are density and pressure of the cosmological matter-distribution of curved spacetime. We had to introduce the suffix $|_F$ to distinguish them from the analogous fields of our test distribution in flat spacetime. In (2)–(3) dots mean derivatives with respect to the coordinate time.

The fact that equations (2)–(3) can be formally obtained in a 5-d special relativistic framework is not completely surprising, since the full 5-d hydrodynamical system gives us free parameters to handle with, but it is also non trivial, since Friedmann equations are Einstein's gravitational equations, while we are working in a flat-spacetime, with no gravitation.

The sgnificative idea then is that 5-d special relativistic hydrodynamics can simulate, at least to some extent, general relativistic cosmology.

2. 5-D SPECIAL RELATIVISTIC HYDRODYNAMICS

Let us briefly recall the notations and hypothesis introduced in Gemelli (2006), with some important completion.

Let us consider a 5-dimensional flat manifold $\mathcal{M}_5$. We leave for the moment the possibility for the signature to be $- + + + -$ or $- + + + +$ as well; to this aim we will introduce in the equations a scalar ϵ which can assume the values $+1$ or -1. Let capital latin indices run from 0 to 4 and greek letters run from 0 to 3. We choose orthonormal coordinates, such that the 5-dimensional line element is:

$$g_{AB}dx^A dx^B = -dt^2 + dx^2 + dy^2 + dz^2 + \epsilon d\xi^2 \tag{4}$$

where we have denoted $t = x^0$, $x = x^1$, $y = x^2$, $z = x^3$ and $\xi = x^4$.

In the case of cosmological special relativity we have $\epsilon = -1$ and $\xi = v$, where v is the receding velocity of galaxies. In fact if $\epsilon = -1$ we may equivalently being considering the 5-d extension of cosmological special relativity or the flat-spacetime specialization of Carmeli's general relativistic brane world theory (Carmeli, 2002). In practice we set both the speed of light in vacuo $c = 1$ and the Hubble-Carmeli constant (analogue to c for the fifth dimension) $H_0^{-1} = 1$.

Let us now define 5-d perfect hydrodynamics in an axiomatic way, by analogy with 4-d hydrodynamics, and then study the 4-d consequences of such choice. Let T^{AB} be a 5-d conserved perfect fluid stress-energy tensor, i.e. such that:

$$\partial_A T^{AB} = 0 \tag{5}$$

and that:

$$T^{AB} = (M + Q)V^A V^B + Q g^{AB} \tag{6}$$

Here the "thermodynamical" fields M and Q are supposed to be defined and regular in a 5-d "world tube" generated by a geometrical congruence of lines tangent to V. Let s be a privileged parameter along one of such lines, which we denote by ℓ, with parametric equations $x^A = X^A(s)$; we thus have, along ℓ: $M = M(s)$, $Q = Q(s)$, and:

$$V^A = \frac{dx^A}{ds}. \tag{7}$$

Note that in Gemelli (2006) the symbol R was used in place of M; here we instead deserve R for a more appropriate use (see Section 4). Now let us introduce the following splitting:

$$V^A = W^A + \mu \Xi^A \tag{8}$$

where $\Xi^A = \delta^A{}_\xi$ is the direction of the fifth dimension and μ a free parameter, equivalent to the square of V, as we are going to see. We have: $V^\alpha = W^\alpha = dx^\alpha/ds$ and $V^\xi = \mu = d\xi/ds$.

Let us moreover denote by a star the derivative with respect to s and by prime the derivative with respect to ξ, so that we have:

$$(\ )^\star = V^A \partial_A = W^\alpha \partial_\alpha + \mu(\ )'. \tag{9}$$

We also have: $V^A = (X^A)^\star$, $\xi^\star = \mu$. Let us denote $E = T^{\xi\xi}$. Now system (5) equivalently reads:

$$\begin{aligned} \partial_\alpha T^{\alpha\xi} + E' &= 0 \\ \partial_\alpha T^{\alpha\beta} + (T^{\xi\beta})' &= 0 \end{aligned} \tag{10}$$

Recall that for an ordinary perfect fluid the 4-d system of hydrodynamics is (Lichnerowicz, 1967; Anile, 1989; Lichnerowicz, 1994):

$$\begin{aligned} &\partial_\alpha(rU^\alpha) = 0 \\ &\partial_\alpha[(\rho + p)U^\alpha U^\beta + pg^{\alpha\beta}] = 0 \end{aligned} \tag{11}$$

where r is the baryon number, ρ is the density, p is the pressure and U the unit 4-velocity vector, and $g^{\alpha\beta}$ the Minkowski metric. We have by definition: $\rho = r(1 + \mathcal{E})$ where $\mathcal{E}$ is the internal energy. We suppose that an equation of state of the kind $p = p(r, S)$ is given and that the thermodynamical principle (1) holds.

To have a significative match between (10) and (12) we have to suppose:

$$\begin{aligned} &\mu(M + Q)W^\alpha = rU^\alpha \\ &(M + Q)W^\alpha W^\beta + Qg^{\alpha\beta} = (\rho + p)U^\alpha U^\beta + pg^{\alpha\beta} \end{aligned} \tag{12}$$

One thus necessarily finds:

$$Q = p, \quad M = \frac{r^2}{\mu^2(\rho + p)} - p \tag{13}$$

and consequently:

$$W^\alpha = \mu \frac{\rho + p}{r} U^\alpha. \tag{14}$$

In relativistiv hydrodynamics the variable $f = (\rho + p)/r$ is called fluid index, and it is $f = i + 1$, where i is the specific enthalpy (see Lichnerowicz, 1994 p. 99). We then have:

$$W^\alpha = \mu f U^\alpha. \tag{15}$$

We also have:

$$V^2 = V^A V_A = \mu^2[\epsilon - f^2] \tag{16}$$

By introducing the symbol V^2 we implicitedly assume $V_A V^A > 0$. Even if we could do without such hypothesys, for the moment we prefer to work with $V_A V^A > 0$ for the sake of simplicity. We see from (16) that the parameter μ is equivalent to the square of V, as said before. We leave μ as a free parameter for the moment. For the sake of brevity we denote by a dot derivative with respect to proper time, i.e.: $(\ )^{\cdot} = U^{\alpha}\partial_{\alpha}$; this should not be confused with derivative with respect to coordinate time, as it is instead in (2)–(3). From (9) we then have:

$$(\ )^{\star} = \mu[f(\ )^{\cdot} + (\ )'] \tag{17}$$

Now let us apply (17) to ξ and compare with $\xi^{\star} = \mu$; we are led to the following identity:

$$f\dot{\xi} = 0 \tag{18}$$

We discard for the moment the singular situation $f = 0$ (otherwise $\rho + p = 0$) and conclude from (18) that $\dot{\xi} = 0$. Our hydrodynamical system now reads as follows:

$$\begin{aligned} &\partial_{\alpha}(rU^{\alpha}) + E' = 0 \\ &\partial_{\alpha}T^{\alpha\beta} + (rU^{\beta})' = 0 \end{aligned} \tag{19}$$

In particular, the function $-E'$ is interpretable as the source of particle production.

3. COSMOLOGICAL PARTICLE PRODUCTION

Let us now introduce in (20) the expression of the 4-d component of the stress-energy tensor:

$$T^{\alpha\beta} = (\rho + p)U^{\alpha}U^{\beta} + pg^{\alpha\beta} \tag{20}$$

and split the system with respect to U and the orthogonal local rest space (Jantzen *et al.*, 1992); we have:

$$\begin{aligned} &\dot{r} + r\partial_{\alpha}U^{\alpha} + E' = 0 \\ &\dot{\rho} + (\rho + p)\partial_{\alpha}U^{\alpha} + r' = 0 \\ &(\rho + p)\dot{U}^{\beta} + r(U^{\beta})' + \partial^{\beta}p + U^{\beta}\dot{p} = 0 \end{aligned} \tag{21}$$

Note that if we remove from (22) all the terms with a prime, i.e. we consider in a sense the ordinary 4-d situation, in wich all fields are independent on ξ, we obtain nothing but the ordinary 4-d hydrodynamical system:

$$\begin{aligned} &\dot{r} + r\partial_{\alpha}U^{\alpha} = 0 \\ &\dot{\rho} + (\rho + p)\partial_{\alpha}U^{\alpha} \\ &(\rho + p)\dot{U}^{\beta} + \partial^{\beta}p + U^{\beta}\dot{p} = 0 \end{aligned} \tag{22}$$

From $(22)_1$ and $(22)_2$ we have:

$$\dot{\rho} - f(\dot{r} + E') + r' = 0 \tag{23}$$

Now since we have by definition (Gemelli, 2006):

$$\rho = r(1 + \mathcal{E}) \tag{24}$$

where $\mathcal{E}$ is the internal energy, from the thermodynamical principle (1) we have:

$$d\rho = f dr + rT dS \tag{25}$$

so that from (23) we have the following relation for E':

$$E' = f^{-1}(rT\dot{S} + r') \tag{26}$$

However, by definition we have (Gemelli, 2006):

$$E = \frac{r^2}{\rho + p} + \epsilon p = rf^{-1} + \epsilon p \tag{27}$$

and thus, from comparison of (23) and (27) we have:

$$rf^{-1}T(\dot{S} + f^{-1}S') = (\epsilon - f^{-2})p' \tag{28}$$

or equivalently, from (9):

$$r^2 f^{-2}\mu^{-1}TS^{\star} = (\epsilon - f^{-2})p' \tag{29}$$

From (29) we have that if all fields are independent on ξ, like in ordinary 4-d hydrodynamics, we have $\dot{S} = 0$, which in fact is a well known consequence of system (23).

From (29) it is also possible to conclude that particle production is absent if the fluid is isentropic, i.e. if $dS = 0$ [and the equation of state consequently reduces to $p = p(r)$] then $E' = 0$. In fact if $dS = 0$ we have:

$$(\epsilon - f^{-2})p' = 0 \tag{30}$$

and therefore there are two possible situations: $p' = 0$ or $\epsilon = f^{-2}$.

If $p' = 0$ then from the equation of state we also have $r' = \rho' = 0$ and consequently $E' = 0$.

If instead $\epsilon = f^{-2}$, then we must have $\epsilon = +1$ and $r^2 = (\rho + p)^2$. We consequently have:

$$rdr = (\rho + p)(d\rho + dp) \tag{31}$$

Now from (1) if $dS = 0$ we have $d\rho = r^{-1}(\rho + p)dr$ so that from (31) we have:

$$(\rho + p)dp = 0 \tag{32}$$

Excluding the singular case $\rho + p = 0$ we conclude $dp = 0$ and consequently $dr = d\rho = 0$, which implies $dE = 0$ and thus again $E' = 0$.

Thus in any case the source of particle production vanishes if $dS = 0$.

4. FRIEDMANN COSMOLOGY

Let us consider the general case $dS \not\equiv 0$, i.e. with a possibly nonvanishing source of particle production, and let us turn to the original 5-d system. In relativistic hydrodynamics the variable $\mathcal{T} = f/r$ is called dynamical volume (see Lichnerowicz, 1994, p. 99). Let $\Phi = (\mu^2\mathcal{T})^{-1}$. From (13) we then have

$$M + Q = \Phi \tag{33}$$

It is a useful idea, on physical terms, to imagine that the dynamical volume should be proportional to the cube of a parameter R, representing a "typical length," and that therefore our variable Φ should be proportional to R^{-3}; we will introduce this hypothesis later on.

From (5) and (6) we then obtain the following general form of the 5-d system:

$$\Phi^\star V^B + \Phi(V^B)^\star + \Phi\partial_A V^A V^B + \partial^B p = 0 \tag{34}$$

We now are going to consider some useful consequences of system (34).

By multiplying (34) by X_B we have:

$$\Phi X_B(V^B)^\star + (\phi^\star + \Phi\partial_A V^A)X_B V^B + X^B\partial_B p = 0 \tag{35}$$

By multiplying (34) by V_B we have:

$$\Phi V_B(V^B)^\star + (\Phi^\star + \Phi\partial_A V^A)V_B V^B + p^\star = 0 \tag{36}$$

Finally, by taking the 5-d divergence of (34), i.e. in practice by multiplying it by ∂_B, we have:

$$\begin{aligned}\Phi^{\star\star} + 2\phi^\star\partial_A V^A + (V^B)^\star\partial_B\Phi + \Phi[\partial_B(V^B)^\star + (\partial_B V^B)^\star]\\ + \Phi(\partial_A V^A)^2 + \partial_A\partial^A p = 0\end{aligned} \tag{37}$$

Now, since we have $V^B = (X^B)^\star$ we have:

$$\begin{aligned}\partial X_B V^B &= (X^B X_B)^\star/2\\ \partial X_B(V^B)^\star &= (X^B X_B)^{\star\star}/2 - V_B V^B\\ \partial V_B(V^B)^\star &= (V_B V^B)^\star/2\end{aligned} \tag{38}$$

Moreover, since $\Phi = \Phi(s)$, we write: $(V^B)^\star\partial_B\Phi = \Phi^\star\partial_A V^A$. In fact:

$$\Phi^\star(V^B)^\star\frac{ds}{dX^B} = \Phi^\star\frac{(V^B)^\star}{(X^B)^\star}$$

and we also have:

$$\frac{(V^B)^\star}{(X^B)^\star} = \frac{dV^B}{ds}\frac{ds}{dX^B} = \frac{dV^B}{dX^B}$$

Therefore, denoting, for the sake of brevity: $X^2 = X_B X^B$, $\partial_X = X^B \partial_B$, $\Delta = \partial_A \partial^A$ and $\nabla V = \partial_A V^A$ we have that (35)–(37) take the following form:

$$(1/2)\Phi[(X^2)^{\star\star} - V^2] + (1/2)(\Phi^\star + \Phi\nabla V)(X^2)^\star + \partial_X p = 0 \tag{39}$$

$$\Phi^\star V^2 + (1/2)\Phi(V^2)^\star + \Phi\nabla V V^2 + p^\star = 0 \tag{40}$$

$$\Phi^{\star\star} + 3\phi^\star \nabla V + \Phi[\partial_A (V^A)^\star + \nabla V^\star] + \Phi\nabla V^2 + \Delta p = 0 \tag{41}$$

Note that, with our use of the symbol X^2 we again implicitedly assume, for the sake of simplicity, $X_A X^A > 0$, which is restrictive, since $X^A X_A = x_i x^i - t^2 - \xi^2$ in general could be non positive. However we may be considering $t = 0$ (present time) and "large distances" in a sense.

Now let us introduce the typical length parameter R, in a crude and simple way, i.e. by taking $X^2 = R^2$ and $\Phi = R^{-3}$. This R should not be confused with the Ricci scalar of general relativity. It also should not be taken as implying the existence of a physical boundary. From (39)–(41) we have respectively:

$$\Phi[(R^\star)^2 + RR^{\star\star} - V^2/2] + (-3\Phi R^\star/R + \Phi\nabla V)RR^\star + \partial_X p = 0 \tag{42}$$

$$-3\Phi V^2 R^\star/R + (1/2)\Phi(V^2)^\star + \Phi\nabla V V^2 + p^\star = 0 \tag{43}$$

$$\begin{aligned} &- \Phi[3R^\star/R - 12(R^\star/R)^2 + 9\nabla V R^\star/R] + \Phi[(\nabla V)^\star + \nabla(V^\star)] \\ &+ \Phi\nabla V^2 + \Delta p = 0 \end{aligned} \tag{44}$$

Let us now suppose the cosmological fluid has a quasi-isotropic and slow-varying pressure. This rough hypothesis could certainly be replaced by some more general extimate on the dependence of dp on R. For example the rest of our treatment would be substantially unchanged if we would assume $dp \propto R^{-2}$ and $\Delta p \propto R^{-3}$, but in absence of a concrete physical basis for such extimates, we prefer to simply neglect all terms depending on the derivatives of the pressure. Assuming constant or quasi-constant pressure still does not mean assuming a trivial thermodynamics unless one additionally assumes $dS = 0$. Note moreover that in our formal recovering of the Friedmann equations (2)-(3), the thermodynamical variables ρ_F and p_F will be different than our special relativistic analogoues: they will depend on V^2, $\nabla(V)^\star$ and $(\nabla V)^\star$ as well as on ρ and p. In practice constant p doesn not mean constant p_F. This leaves us a certain freedom of choice of hypothesys on the evolution of the special relativistic test-fluid. Now from (43) we have:

$$\nabla V = 3\frac{R^\star}{R} - \alpha \tag{45}$$

where we have denoted:

$$\alpha = \frac{1}{2}\frac{(V^2)^\star}{V^2} \tag{46}$$

Replacing ∇V by (45) in (42) and (43) we have:

$$\frac{R^{\star\star}}{R} + \left(\frac{R^{\star}}{R}\right)^2 - \alpha\frac{R^{\star}}{R} - \frac{1}{2}\frac{V^2}{R^2} = 0 \tag{47}$$

$$\frac{R^{\star\star}}{R} + 2\left(\frac{R^{\star}}{R}\right)^2 - \alpha\frac{R^{\star}}{R} - \frac{1}{3}(\beta + \alpha^2) = 0 \tag{48}$$

where we have denoted:

$$\beta = (\nabla V)^{\star} + \nabla(V^{\star}) \tag{49}$$

Taking (48) minus (47) we then have:

$$\left(\frac{R^{\star}}{R}\right)^2 = \frac{1}{3}(\beta + \alpha^2) - \frac{1}{2}\frac{V^2}{R^2} \tag{50}$$

We recognize the same structure of the Friedmann equation (2). The corrispondence is only formal, since we have to somehow identify time derivative with derivative with respect to s, and the scale factor of Friedmann cosmology (which comes from the metric of the curved 4-d spacetime) with our "typical length." Yet such correspondance is significant. We have that (50) reduces to (2) if:

$$\begin{aligned} \chi\rho_F + \Lambda &= \beta + \alpha^2 \\ k &= V^2/2 \end{aligned} \tag{51}$$

In particular our working hypothesis $V_A V^A > 0$ leads to $k = 1$. Exact match with the value $+1$ is not a problem since we still can handle with the free parameter μ: see (16). But it is clear that in general the sign of k is determined by that of $V_A V^A$: we have $k = 0$ if $V^2 = 0$ and $k = -1$ if $V_A V^A = -V^2 < 0$. In particular, if $\epsilon = -1$ (Carmelian relativity) from (30) we necessarily have $k = -1$.

Now replacing $R^{\star}/R$ from (50) in (47), we have:

$$\frac{R^{\star\star}}{R} = -\frac{1}{3}(\beta + \alpha^2) + \frac{V^2}{R^2} + \alpha\sqrt{\frac{1}{3}(\beta + \alpha^2) - \frac{V^2}{2R^2}} \tag{52}$$

By power series expansion in terms of R^{-1} we have:

$$\sqrt{\frac{1}{3}(\beta + \alpha^2) - \frac{V^2}{2R^2}} = \frac{1}{3}\sqrt{3(\beta + \alpha^2)} - \frac{\sqrt{3(\beta + \alpha^2)}}{4(\beta + \alpha^2)}\frac{V^2}{R^2} + O\left(\frac{1}{R^4}\right) \tag{53}$$

Thus, dropping terms of higher orders, we have from (52):

$$\frac{R^{\star\star}}{R} = \frac{1}{3}[\sqrt{3\beta + 3\alpha^2} - (\beta + \alpha^2)] + \left(1 - \frac{\sqrt{3(\beta + \alpha^2)}}{4(\beta + \alpha^2)}\right)\frac{V^2}{R^2} \tag{54}$$

This appears to introduce a correction term of order R^{-2} to our analogue to Friedmann equation (3). However we can still match the terms of order zero if:

$$\frac{\chi}{6}(\rho_F + 3P_F) + \frac{\lambda}{3} = \frac{1}{3}\left[\sqrt{3\beta + 3\alpha^2} - (\beta + \alpha^2)\right] \tag{55}$$

A solution to system (51)–(55) is the following:

$$\chi\rho = \beta + \alpha^2 \chi p = (1/3)[\beta + \alpha^2 - 2\sqrt{\beta + \alpha^2}] \tag{56}$$

The possible additional condition: $\beta = 3/16 - \alpha^2$ lets the R^{-2} correction vanish, so that both Friedmann equations (2)–(3) are formally recovered, but leads to constant values for ρ_F and p_F:

$$\chi\rho_F = 3/16 \quad \chi p_F = -7/16 \tag{57}$$

Thus in this case we have $p_F < 0$.

Negative pressure cannot be discarded in cosmology and astrophysics (see e.g. Bonnor (1960); Künzle (1967); Wesson (1986); Ebert (1989); Katz and Lyndel-Bell (1991)) and even in some hydrodynamical problems (involving turbolence and moving boundaries: see e.g. Manarini (1948); Greenhow and Moyo (1997); Lifschitz (1998)). Here it is even less surprising, since we are dealing with cosmological particle production. However, our source $-E'$ of particle production is actually independent on the value or sign of p_F, i.e. the particle production mechanism considered in Sections 2 and 3 does not need $p_F < 0$ nor $p < 0$.

REFERENCES

Anile, A. M. (1989). *Relativistic fluids and magneto - fluids* Cambridge univ. press, Cambridge - New York.

Bonnor, W. B. (1960). *Journal of Mathematics and Mechanics* **9**, 439.

Carmeli, M. (1995). *Foundations of Physics* **25**, 1029.

Carmeli, M. (1996). *Foundations of Physics* **26**, 413.

Carmeli, M. (2002). *Cosmological special relativity. The large scale structure of space, time and velocity* (second edition) World Scientific Singapore.

Cissoko, M. (1998). *General Relativity and Gravitation* **30**, 521.

de Campos, M. (2002). *General Relativity and Gravitation* **34**, 1393.

Ebert, R. (1989). In Proceedings of the Fifth Marcel Grossman Meeting on General Relativity, Part A, B, (Perth, 1988), World Sci. Publishing, Teaneck, NJ.

Gemelli, G. (2006). *International Journal of Theoretical Physics* ISSN: 0020-7748 (Paper) 1572-9575 (Online) DOI: 10.1007/s10773-006-9187-z.

Greenhow, M. and Moyo, S. (1997). *Philosophical Transactions of the Royal Society of London. Series A* **355**, 551.

Jantzen, R.T., Carini, P., and Bini, D. (1992). *Annals of Physics* **215**, 1.

Künzle, H. P. (1967). *Proceedings of the Royal Society Series A* **297**, 244.

Katz, J. and Lyndel-Bell, D. (1991). *Classical and Quantum Gravity* **8**, 2231.

Lichnerowicz, A. (1967). *Relativistic hydrodynamics and magneto—hydrodynamics* Benjamin, New York.

Lichnerowicz, A. (1994). *Magnetohydrodynamics: waves and shock waves in curved space-time*, Mathematical physics studies vol. 14, Kluwer academic publishers, Dordrecht-Boston-London.

Lifschitz, A. (1998). In Differential and integral operators (Regensburg, 1995), *Operator Theory: Advances and Applications* **102**, 97, Birkhäuser, Basel.

Manarini, M. (1948). *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat.* **4**, 427.

Prigogine, I., Geheniau, J., Gunzig, E., and Nardone, P. (1989). *General Relativity and Gravitation* **21**, 767.

Wesson, P. S. (1986). *Physical Review D* **34**, 3925.

Wesson, P. S. (1999). *Space, time, matter: modern Kaluza-Klein theory* World Scientific, Singapore.